Extremal lattices.

Extremal lattices

Theorem (Siegel) Let $\Lambda \subset \mathbb{R}^d$ be an even unimodular lattice. Then the lenth ℓ_{min} of the shortest non-zero vector of Λ satisfies:

$$\ell_{min}^2 \leq 2\left\lfloor \frac{d}{24} \right\rfloor + 2.$$

Definition: An even unimodular lattice $\Lambda \subset \mathbb{R}^d$ is called *extremal* if this bound is attained, i.e.

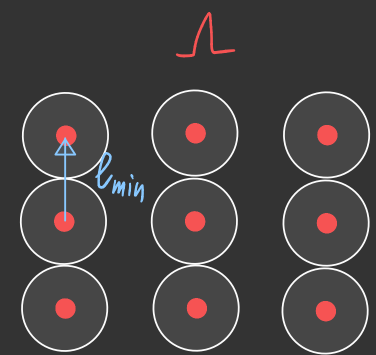
$$\ell_{min}^2 = 2\left\lfloor \frac{d}{24} \right\rfloor + 2.$$

The ball with center x and radius r

$$B(x, r) := \{ y \in \mathbb{R}^d \mid |x-y| < r \}.$$

The sphere packing associated to a lattice $\Lambda \subset \mathbb{R}^d$

$$\mathcal{P}_\Lambda := \bigcup_{\ell \in \Lambda} B\left(\ell, \frac{\ell_{min}}{2}\right)$$



The density of $\mathcal{P}_\Lambda$

$$\Delta_{\mathcal{P}} := \frac{\operatorname{vol} B\left(0, \frac{\ell_{min}}{2}\right)}{\operatorname{vol}\left(\mathbb{R}^d / \Lambda\right)}$$

Examples:

(1) $d = 8$

$$\ell_{min}^2 \leq 2\left\lfloor \frac{d}{24} \right\rfloor + 2 = 2$$

E_8 is an extremal lattice in dim 8.

(2) $d = 24$

$$\ell_{min}^2 \leq 2 \cdot 1 + 2 = 4$$

The unique extremal lattice in $\mathbb{R}^{24}$ is the Leech lattice.

Extremal modular forms

Let $k \in 4\mathbb{Z}_{>0}$ $\quad k = d/2$

Theorem: There exists a unique modular form $f \in M_k(\Gamma_1)$ such that its Fourier coefficients satisfy:

$$c_f(0) = 1, \quad c_f(1) = c_f(2) = \dots = c_f(\lfloor \tfrac{k}{12} \rfloor) = 0.$$

Recall: $$f(z) = \sum_{n=0}^{\infty} c_f(n) e^{2\pi i n z}.$$

This modular form f is called the extremal modular form.

Moreover, $c_f(\lfloor \frac{k}{12} \rfloor + 1) > 0.$

$k = 4s$ $\qquad$ $s := k/4 \in \mathbb{Z}_{>0}$

Proof:

$M_{4s}(\Gamma_1)$ has the following basis:

$$E_4^s = 1 + 240s \cdot e^{2\pi i z} + \dots$$

$$E_4^{s-3} \cdot \Delta = e^{2\pi i z} + \dots$$

$$E_4^{s-6} \cdot \Delta^2 = e^{4\pi i z} + \dots$$

$$\vdots$$

$$E_4^{s-3t} \cdot \Delta^t = e^{t \cdot 2\pi i z} + \dots$$

$$t = \left\lfloor \frac{k}{12} \right\rfloor = \left\lfloor \frac{s}{3} \right\rfloor$$

It is easy to see from this basis that the extremal modular form is unique.

Set $q = e^{2\pi i z}$ and $C := c_f(t+1)$. $s = s_0 + 3t$

We have $f = 1 + C \cdot q^{t+1} + O(q^{t+2})$.

Notations: $s = \frac{k}{4}$, $t = \lfloor \frac{s}{3} \rfloor$, $s = 3t + s_0$ for some $s_0 \in \{0,1,2\}$.

Claim:

☆ $$C = \frac{t + s_0/3}{t+1} \cdot \text{coefficient}_{q^{-1}} \left[\frac{\frac{d}{dq}(E_4^3)}{E_4^{s_0} \cdot \Delta^{t+1}} \right]$$

Laurent series in variable q

Claim implies that $C > 0$.

$E_4 = 1 + \sum c_{E_4}(n) q^n$, $c_{E_4}(n) > 0$

$$\frac{d}{dq}(E_4^3) \cdot E_4^{-s_0} = \underbrace{3 \cdot \frac{d}{dq} E_4 \cdot E_4^{2-s_0}}$$ $2 - s_0 \geq 0$

this function has positive coefficients in q-expansion

$$\frac{1}{\Delta} = q^{-1} \left(\prod_{n=1}^{\infty} (1 - q^n)^{-1} \right)^{24} = q^{-1} \left(\prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \ldots) \right)^{24}$$

Therefore $C > 0$

Proof of the claim ☆:

We have

$$f = E_4^{s} + \alpha_1 \cdot E_4^{s-3} \cdot \Delta + \alpha_2 E_4^{s-6} \cdot \Delta^2 + \dots + \alpha_t \cdot E_4^{s-3t} \cdot \Delta^t$$

recall: $s = s_0 + 3t$

$$f = E_4^{s_0}\left(\frac{E_4^{3t}}{\Delta^t} + \alpha_1 \frac{E_4^{3(t-1)}}{\Delta^{t-1}} + \dots + \alpha_t \right)\Delta^t$$

$j = \frac{E_4^3}{\Delta}$ $\quad E := E_4^3$ $\quad P(X) = X^t + \alpha_1 X^{t-1} + \dots + \alpha_t$

We can write the extremal modular form f as

$$f = E_4^{s_0} \cdot \Delta^t \cdot P\left(\frac{E}{\Delta}\right) = E_4^{s_0} \cdot \Delta^t \cdot P(j)$$

recall: $f = 1 + C \cdot q^{t+1} + \underline{O}(q^{t+2})$

$$P(j) E_4^{s_0} \cdot \Delta^t = 1 + C \cdot q^{t+1} + \underline{O}(q^{t+2}) \Rightarrow P(j) = \frac{1}{E_4^{s_0} \cdot \Delta^t} + Cq + \underline{O}(q^2)$$

Recall from the last slide: $P(j) = \frac{1}{E_4^{s_0} \cdot \Delta^t} + Cq + \underline{O}(q^2)$ (☺)

We have: $E_4 = 1 + \underline{O}(q)$, $\Delta = q + \underline{O}(q^2)$, $j = \frac{E_4^3}{\Delta} = q^{-1} + \underline{O}(1)$

We make the change of variables $w := \frac{1}{j(q)}$ $\quad w = q + \underline{O}(q^2)$

we have: $E_4(w) = 1 + \underline{O}(w)$, $\Delta(w) = w + \underline{O}(w^2)$, $j = \frac{1}{w}$

Now (☺) implies

$$\frac{1}{E_4^{s_0}(w) \cdot \Delta^t(w)} = \underbrace{P\left(\frac{1}{w}\right) - C \cdot w + \underline{O}(w^2)}_{\text{Laurent expansion with respect to the variable } w.}$$

This identity allows us to compute C as a residue of a meromorphic differential:

$$\Rightarrow \; -C = \frac{-1}{2\pi i} \operatorname{res}_{w=0} \left[\frac{1}{E_4^{s_0}(w)\, \Delta^t(w)} \, \frac{dw}{w^2} \right].$$

A final step is to make the change of variables $w \mapsto q$

We recall that for a meromorphic differential $\Omega = F(z)dz$

$$\operatorname{Res}_{z=0}(\Omega) = -2\pi i \ \text{Coefficient}_{z^{-1}}(F(z))$$

for any holomorphic coordinate z in a nbh of 0.

We have

$$\frac{1}{E_4^{s_0}(w)\Delta^t(w)} \frac{dw}{w^2} = \frac{\frac{dw}{dq} w^{-2}}{E_4^{s_0}(q)\Delta^t(q)} dq$$

Now we make the final computation and show that

$$\operatorname{res}_{q=0}\left[\frac{\frac{dw}{dq} w^{-2}}{E_4^{s_0}(q)\Delta^t(q)} dq\right] = \operatorname{res}_{q=0}\left[\frac{\frac{d}{dq}(E_4^3)}{E_4^{s_0}\cdot\Delta^{t+1}}\right]$$

$$\mathcal{C} = -\mathrm{Res}_{q=0}\left[\frac{1}{E_4^{s_0}\Delta^t}\,\frac{d\left(\frac{\Delta}{E_4^3}\right)}{\left(\frac{\Delta}{E_4^3}\right)^2}\right] =$$

Set $r = \frac{s_0}{3}$

$$E_4^{-s_0+6}\cdot\Delta^{-t-2}\,\frac{(d\Delta\cdot E - \Delta\cdot dE)}{E^2} = E^{-r}\cdot\Delta^{-t-2}(d\Delta\cdot E - \Delta\cdot dE)$$

$$= \mathrm{res}_{q=0}\left(E^{-r+1}\Delta^{-t-2}d\Delta - E^{-r}\Delta^{-t-1}dE\right)$$

Recall that for a meromorphic function F we have $\mathrm{res}(dF) = 0$.

$$0 = \mathrm{res}\left[d\left(E^{-r+1}\Delta^{-t-1}\right)\right] = \mathrm{res}\left[(-r+1)E^{-r}\Delta^{-t-1}dE + (-t-1)E^{-r+1}\Delta^{-t-2}d\Delta\right]$$

$$\Downarrow$$

$$\mathrm{res}_{q=0}\left(E^{-r+1}\Delta^{-t-2}d\Delta\right) = \mathrm{res}_{q=0}\left(\frac{1-r}{t+1}E^{-r}\cdot\Delta^{-t-1}dE\right)$$

$$\mathcal{C} = \mathrm{Res}_{q=0}\left(-\left(\frac{1-r}{t+1} - 1\right)E^{-r}\cdot\Delta^{-t-1}dE\right)$$

this proves the claim ☆ and finishes the proof ▨

Packing density of extremal lattices.

Let $\Lambda \subset \mathbb{R}^d$ be an extremal lattice

Then $$\bigcup_{\ell \in \Lambda} B\left(\ell, \frac{\sqrt{\lfloor \frac{d}{24} \rfloor + 1}}{\sqrt{2}}\right)$$

Euclidean balls with center and radius

$$\text{packing density} := \frac{\text{volume of balls}}{\text{volume of the space}} = \frac{\mathrm{vol}\,(B(0, r_d))}{\mathrm{vol}\,(\mathbb{R}^d/\Lambda)} \approx$$

by Stirling

$$\approx (\pi e/24)^{d/2 + \bar{o}(d)} \approx 2^{-0.745 d + \bar{o}(d)}$$

The sphere packing constant $\Delta_d := \sup_{\mathcal{P} \subset \mathbb{R}^d} \Delta_{\mathcal{P}}$

Minkowski - Hlawka bound

$$\Delta_d \geq 2^{-d + \bar{o}(d)}$$

extremal lattices

$$2^{-0.745 d + \bar{o}(d)}$$

Kabatianski - Levenstein Bound

$$\Delta_d \leq 2^{-0.599 d + \bar{o}(d)}$$

Theorem (Mallows–Odlyzko–Sloane)

There are only finitely many non-isomorphic extremal even unimodular lattices

This beautiful proof is beyond the scope of our course

Idea of the proof:

Let $f_k \in M_k(\Gamma_1)$ be the extremal modular form

$$f_k = 1 + c_k q^{\lfloor \frac{k}{12} \rfloor + 1} + \tilde{c}_k q^{\lfloor \frac{k}{12} \rfloor + 2} + \ldots$$

$\underline{\tilde{c}_k < 0}$ for $k \gg 0$.

$k > 2000$

$d > 4000$